Goldstone bosons

- P&S 11.1, 20.1, 21.2
- Schwartz 28

We've seen how, in the case of SSB of a $U(1)$ global symmetry we got a Goldstone boson. We have

"1 broken generator" = "1 massless mode"

For the $SO(N)$ case, the potential induced a SSB with $SO(N) \to SO(N-1)$, with $N-1$ massless modes. The number of "broken" generators is

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1$$

We can show that, indeed, in general the number of massless modes is given by the number of "broken" generators.

For a generic group $\mathcal{G}$ spontaneously broken, it is always possible to choose the T^A's of the group such that

$$T^A = \{ T^a , T^{\hat{a}} \}$$

"unbroken" $T^a \vec{v} = 0$

"broken" $T^{\hat{a}} \vec{v} \neq 0$

The unbroken ones form a subalgebra:

$$[T^a, T^b] \vec{v} = 0 = i f^{abc} \underbrace{T^c \vec{v}}_{=0} + i f^{ab\hat{c}} \underbrace{T^{\hat{c}} \vec{v}}_{\neq 0}$$

$$\Rightarrow f^{ab\hat{c}} = 0.$$

$$\Rightarrow [T^a, T^b] = i f^{abc} T^c$$

Therefore $\mathcal{G}$ is spontaneously broken to a subgroup $\mathcal{H}$.

- T^a generate unbroken subalgebra.
- $T^{\hat{a}}$ are the broken generators, $\in \mathcal{G}/\mathcal{H}$

Goldstone theorem: there is a massless mode for each "$\hat{a}$".

$$a = 1, \dots, \dim(\mathcal{H})$$
$$\hat{a} = 1, \dots, \dim(\mathcal{G}) - \dim(\mathcal{H})$$

Proof:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_\alpha \partial^\mu \phi^\alpha - V(\phi)$$

That the potential is invariant under $\mathcal{G}$, which infinitesimally is

$$\phi_\alpha \longrightarrow (e^{i\theta_A T^A})_{\alpha\beta} \phi_\beta \simeq \phi_\alpha + i\theta_A T^A_{\alpha\beta} \phi_\beta$$
$$= \phi_\alpha + \delta\phi_\alpha$$

means that

$$V(\phi + \delta\phi) = V(\phi)$$

$$\rightarrow \frac{\partial V}{\partial \phi_\alpha} \delta\phi_\alpha = 0 = \frac{\partial V}{\partial \phi_\alpha} i\alpha_A T^A_{\alpha\beta} \phi_\beta$$

which is valid $\forall$ $A = 1, \dots, \dim(\mathcal{G})$:

$$\frac{\partial V}{\partial \phi_\alpha} T^A_{\alpha\beta} \phi_\beta = 0$$

By taking a $\partial/\partial\phi_\rho$ derivative,

$$\frac{\partial^2 V}{\partial \phi_\alpha \partial \phi_\beta} T^A_{\alpha\gamma} \phi_\gamma + \frac{\partial V}{\partial \phi_\alpha} T^A_{\alpha\beta} = 0$$

Consider now the vacuum configuration, $\vec{\phi} = \vec{\phi}_{min} = \langle \vec{\phi} \rangle$ being the min. of potential, $\vec{v}$. Then

$$\left.\frac{\partial V}{\partial \phi_\alpha}\right|_{\vec{\phi} = \vec{v}} = 0$$

by definition. So the previous eq gives

$$\left.\frac{\partial^2 V}{\partial \phi_\alpha \partial \phi_\beta}\right|_{\vec{v}} \cdot T^A_{\alpha\gamma} v_\gamma = 0$$

for all A.

The matrix $\partial^2 V / \partial \phi_\alpha \phi_\beta$ is the mass matrix $M^2_{\alpha\beta}$.

If $A = a$, then $T^a v = 0$ and the equation is trivial.

If $A = \hat{a}$, then $T^{\hat{a}} v \equiv v^{\hat{a}} \neq 0$, and

$$M^2_{\beta\alpha} (v^{\hat{a}})_\alpha = 0$$

tells us that each $\hat{a}$ is a null eigenvector.

There are $\dim(\mathcal{G}/\mathcal{H}) = \dim(\mathcal{G}) - \dim(\mathcal{H})$ massless modes: Goldstones

This proof is very intuitive (the potential is flat due to symm and therefore there are massless modes), but it is a tree-level proof.

It is possible to "enhance" it to non-pert. level by introducing the effective potential, you will see this in Advanced QFT.

- Alternative proof:

Any continuous symmetry implies the existence of a conserved current

$$\partial_\mu J^\mu = 0.$$

This defines a charge,

$$Q(t) = \int d^3x \, J^0(\vec{x},t)$$

we usually derive that $\frac{d}{dt} Q(t) = 0$. However, for SSB Q diverges.

Still, we are interested in the commutator of Q, which controls the transf. of a field

$$\phi(x) \to \phi'(x) = e^{i\varepsilon Q}\phi(x)e^{-i\varepsilon Q}$$
$$= \phi(x) + i\varepsilon\,[Q, \phi(x)] + \dots$$

Current cons. implies

$$0 = \int d^3x\,[\partial^\mu J_\mu(\vec{x},t), \phi(0)]$$
$$= \partial^0 \int d^3x\,[J^0(x,t), \phi(0)] + \int d\vec{S}\cdot[\vec{J}(x,t), \phi(0)]$$

For large enough surface, second term vanishes and

$$\frac{d}{dt}[Q(t), \phi(0)] = 0$$

For the broken generators, we have that

$$\langle 0|\,[Q(t), \phi(0)]\,|0\rangle \equiv v \neq 0$$

After inserting a complete set of states,

$$\sum_\alpha (2\pi)^3\delta^3(\vec{p}_\alpha)\Big(\langle 0|J_0(0)|\alpha\rangle\langle\alpha|\phi(0)|0\rangle\, e^{-iE_\alpha t}$$
$$- \langle 0|\phi(0)|\alpha\rangle\langle\alpha|J_0(0)|0\rangle\, e^{iE_\alpha t}\Big) = v.$$

To make it time-indep & nonvanishing, and since positive & negative parts do not cancel, it is only satisfied if

$\exists\ |\alpha\rangle$ s.t. $E_\alpha = 0$ for $\vec{p}_\alpha = 0$.

So massless state. It is such that

$$\langle\alpha|\,\phi(0)\,|0\rangle \neq 0$$

$$\langle 0|\,J_0(0)\,|\alpha\rangle \neq 0$$

- We can use the $U(1)$ case to exemplify these results. We had σ and η as dynamical modes. The current is

$$J_\mu = \left((v+\sigma)\partial_\mu\eta - \partial_\mu(v+\sigma)\,\eta\right)$$

with the associated charge being $(\sigma=0)$

$$Q = \int d^3x\, J^0 = \int d^3x\, v\cdot\dot{\eta}$$

using the canonical comm. rel,

$$[Q, \eta(0)] = -iv$$

and

$$\langle\eta|\,\eta(0)\,|0\rangle \neq 0$$

$$\langle 0|\,J_0(0)|\eta\rangle \neq 0$$

By normalizing $\langle\eta|\eta(0)|0\rangle = 1$, then

$$\int \frac{d^3p}{2p_0}\, \delta^3(p) \Big(\langle 0| J_0(0) |\eta(p)\rangle \langle \eta(p)| \eta(0) |0\rangle - \langle 0| \eta(0) |\eta(p)\rangle \langle \eta(p)| J_0(0) |0\rangle \Big) = -iv$$

satisfied for

$$\langle 0| J_0(0) |\eta(p)\rangle = iv p_0$$

Lorentz covariance implies

$$\langle 0| J_\mu(0) |\eta(p)\rangle = iv p_\mu$$

Notice that the matrix element for the divergence is therefore

$$\langle 0| \partial^\mu J_\mu |\eta\rangle = v m_\eta^2$$

so current conservation implies $m_\eta = 0$.

■ Linear and non-linear σ-models

Let us focus now on the $SU(2)_L \times SU(2)_R$ sigma model.

The reason is that it will be a reasonable model for real-world pions, as we will see later in the course.

Take the Lagrangian

$$\mathcal{L} = \frac{1}{4} tr(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \frac{\mu^2}{4} tr(\Sigma^\dagger \Sigma) - \frac{\lambda}{16}(tr \Sigma^\dagger \Sigma)^2$$

invariant under

$$\Sigma \to L \Sigma R^\dagger \quad ; \quad \Sigma^\dagger \to R^\dagger \Sigma L$$

Writing $\Sigma \equiv \sigma + i \tau \cdot \pi$, so that

$$\frac{1}{2} tr \Sigma^\dagger \Sigma = \sigma^2 + \vec{\pi}^2$$

one gets

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma$$
$$+ \frac{\mu^2}{2} (\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2$$

After EWSB, writing

$$\sigma = v + \tilde{\sigma} \quad , \quad v = \sqrt{\mu^2/\lambda}$$

the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} - \mu^2 \tilde{\sigma}^2$$
$$+ \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \lambda v \tilde{\sigma} (\tilde{\sigma}^2 + \vec{\pi}^2) - \frac{\lambda}{4} (\tilde{\sigma} + \vec{\pi}^2)^2$$

So there is a massless pions and a heavy radial mode.

The vacuum is $\langle\Sigma\rangle \sim v\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and therefore only the diagonal part of $SU(2)_L \times SU(2)_R$ remains unbroken, i.e. when $L = R$,

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V$$

with "V" stands for vectorial. On the original fields, the $SU(2)_V$ acts as

$$\sigma \to \sigma' = \sigma$$
$$\pi^i \to \pi^{i\prime} = \pi^i + \epsilon^{ijk}\theta^j\pi^k$$

so the pion is a triplet and the σ does not transform, as expected since it is the unbroken part of the group.

Under axial transformations, however,

$$\sigma \to \sigma' = \sigma + \theta^i\pi^i$$
$$\pi^i \to \pi^{i\prime} = \pi^i - \theta^i\sigma$$

So 1) σ indeed transforms

2) When $\langle\sigma\rangle = v$, the pion transforms with a shift.

At this point it seems relevant to remark that fields are not physical, as it is explicit in the path integral formulation since they are integration variables.

We can write the above Lagrangian in a "square-root" form

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\,\partial^\mu s - \mu^2 s^2 - \lambda v s^3 - \frac{\lambda}{4} s^4$$
$$+ \frac{1}{2}\left(\frac{v+s}{v}\right)^2\left((\partial_\mu\vec{\varphi})^2 + \frac{(\vec{\varphi}\cdot\partial_\mu\varphi)^2}{v^2-\varphi^2}\right)$$

with the advantage that there are no polynomial interactions for the pions. We have

$$s \equiv \sqrt{(\tilde{\sigma}+v)^2+\vec{\pi}^2} - v = \tilde{\sigma} + \dots$$

$$\vec{\varphi} = \frac{v}{\sqrt{(\tilde{\sigma}+v)^2+\vec{\pi}^2}}\,\vec{\pi} = \vec{\pi} + \dots$$

A more convenient form that also imposes the $\phi^2 = v^2$ constraint and removes the pions from the potential is the exponential form,

$$\Sigma = (v+s)\,U \quad ; \quad U = e^{i\frac{\tau\cdot\pi}{v}}$$

leading to

$$\mathcal{L} = \frac{1}{2}\partial_\mu s \partial^\mu s - \mu^2 s^2 - \lambda v s^3 - \frac{\lambda}{4} s^4$$
$$+ \frac{(v+s)^2}{4} \mathrm{tr}(\partial_\mu U \partial^\mu U)$$

- Representation independence

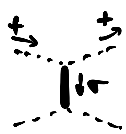
If we don't care about the explicit form of Lagrangians, what should we care about?

Well, physical observables.

The simplest one is the scattering amplitudes of the theory.

We can check that Goldstone scattering is the same in any rep:

1) $\mathcal{L}_I = -\frac{\lambda}{4}(\vec{\pi}^2)^2 - \lambda v \tilde{\sigma} \vec{\pi}^2$,

$\Rightarrow i\mathcal{M}_{\pi^+\pi^0 \to \pi^+\pi^0} =$  $+$ 

$$= -2i\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m_\sigma^2}$$

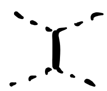
$$= -2i\lambda\left[1 + \frac{2\lambda v^2}{q^2 - 2\lambda v^2}\right] = \frac{i q^2}{v^2} + \ldots$$

where we used $m_\sigma^2 = 2\lambda v^2$.

The amplitude vanishes as $q \to 0$, even though the theory contains no derivatives.

This is because there exists a rep where derivative interactions are explicit!

2) $$\mathcal{L}_I = \frac{1}{2}\frac{(\vec{\varphi}\cdot\partial_\mu\vec{\varphi})^2}{v^2 - \vec{\varphi}^2} + \frac{s}{v}\,\partial_\mu\vec{\varphi}\cdot\partial^\mu\vec{\varphi}$$

Now the  diag contains 4 powers of momenta. So for $q^2 \to 0$ the first term is enough.

$$i\mathcal{M}_{\pi^+\pi^0\to\pi^+\pi^0} = \frac{i q^2}{v^2} + \dots$$

3) $$\mathcal{L}_I = \frac{(v+s)^2}{4}\operatorname{tr}(\partial_\mu U \partial^\mu U) + \dots$$

similar to square root rep.

- This is general: if two fields φ & χ are related by $\varphi = \chi F(\chi)$ with $F(0) = 1$, then physical observables computed with φ and χ are indistinguishible. The proof relies on showing that φ & χ interpolate same one-particle states.